Math 4200
Wednesday November 25
Chapter 5: 5.1-5.2 conformal maps and fractional linear transformations, continued.
We'll begin by finishing the discussion in Monday's notes. Then we'll discuss the algebra and geometry of linear fractional transformations, which are introduced there.

Announcements:

Math 4200-001
Week 14 concepts and homework
5.1-5.2
Due Thursday December 3 at 11:59 p.m.

5.1: 10, 11, 12.
5.2 1, 4a, 6, 7, 9, 10, 17, 24, 26, 33, 34

Continuing the discussion from Monday,

Example Let

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

be a (non-constant) FLT. Use matrix algebra to find a formula for $f^{-1}(z)$.

Corollary Fractional linear transformations are bijections of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. In fact, regarding the Riemann sphere as a *Riemann surface* (see later discussion), it turns out that these all of the only conformal bijections of the Riemann sphere with itself.

Theorem Fractional linear transformations map the set of all circles and lines to itself.

proof: Any circle or line in the $x - y$ plane can be described implicitly as the solution set to an equation

$$(1) \qquad A\left(x^2 + y^2\right) + B\,x + C\,y + D = 0$$

where A, B, C, $D \in \mathbb{R}$ and are not all zero.

We already know that translating or rotating circles (resp. lines) yields circles (resp. lines). So the Theorem holds for the first two transformations below. Show it also holds for inversions, the third transformation.

$$T_1(z) = z + a \quad \text{(translation)}$$

$$T_2(z) = c\,z \quad \text{(rotation-dilation)}$$

$$T_3(z) = \frac{1}{z} \quad \text{(inversion)}$$

convert the solution set of an equation of form (1) into the solutions set of a (different) equation of form (1).

Then show that any fractional linear transformation

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

is a composition of translations, rotation-dilations, and inversions. Hint: Treat $c = 0$, $c \neq 0$ separately. If $c \neq 0$ first do something equivalent to long division to rewrite f.

Notice that

$$f(z) = \frac{z-a}{z-b}\left(\frac{c-b}{c-a}\right)$$

maps

$$a \to 0$$
$$b \to \infty$$
$$c \to 1.$$

Since 3 points uniquely determine particular circles one can use FLT's to map any circle or line to any other circle or line.

Using functions of this form, and their inverses, one can construct FLT's to map triples of points to triples of points:

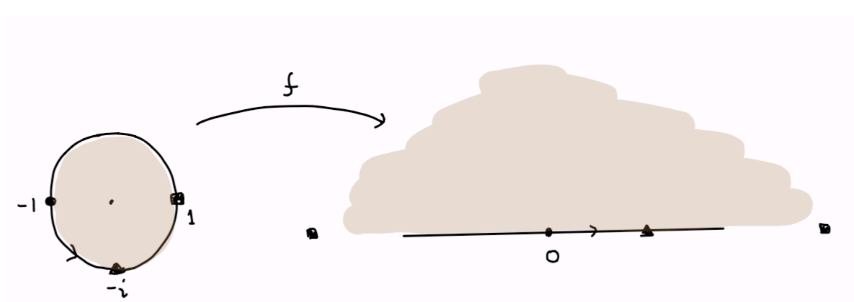
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \to \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

Thus you can map any line or circle to any other line or circle.

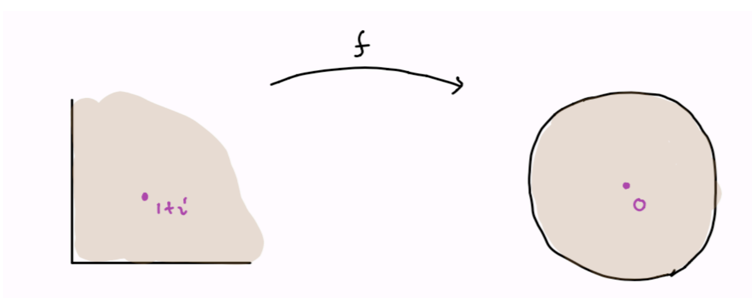
Example Find a FLT from the unit disk to the upper half plane by mapping

$$-1 \to 0$$
$$1 \to \infty$$
$$-i \to 1$$

and making any necessary adjustments. (By magic, once you know the boundary of the disk goes to the real axis, you only have to check that one interior point goes to an interior point, or that the orientation is correct along the boundary, to know that you're mapping the unit disk to the upper half plane instead of the lower half plane. The proof of the magic theorem is an appendix in today's notes.)



Example Find a conformal transformation of the first quadrant to the unit disk, so that the image of $1 + i$ is the origin. How many such conformal transformations are there? It's fine to write your transformation as a composition.



Riemann surfaces: These are special cases of *two-dimensional differentiable manifolds,* in the case that the *transition functions* between atlas pages are all conformal diffeomorphisms. (See Wikipedia.)

Definition A *Riemann surface* S is a topological space S together with an *atlas* consisting of *charts* $\{U_\alpha, \varphi_\alpha\}_{\alpha \in A}$ where

(1) $\bigcup_{\alpha \in A} U_\alpha = S$ and each U_α is open.

(2) Each $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{C}$ is a *homeomorphism.* We can call the sets V_α *pages* of the atlas.

(3) The *transition maps* between parts of the pages of the *atlas* $\varphi_\beta \circ \varphi_\alpha^{-1} : \ \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are all conformal.

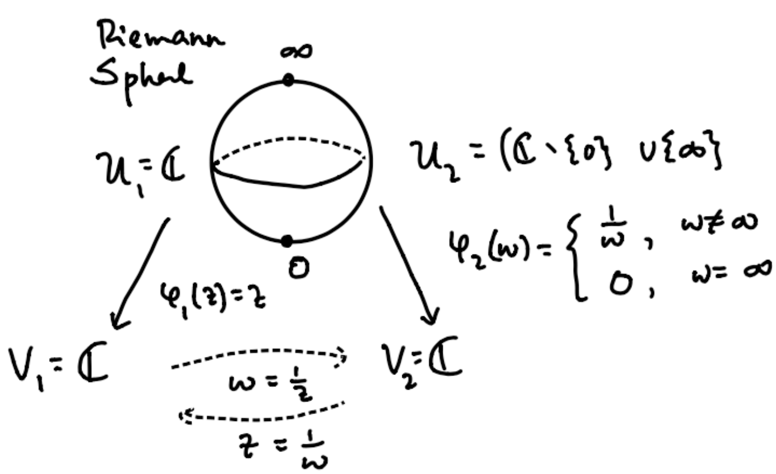
This definition makes sense when you think of what an actual geographical atlas is, along with a few concrete examples including the Riemann sphere:

- The complex plane itself, or any open set in the complex plane is a Riemann surface which has one possible atlas consisting of a single page, with $U = V$ and $\varphi = id$.

- The Riemann sphere $\mathbb{C} \cup \{\infty\}$, which is homeomorphic to the unit sphere in $\mathbb{R}^3$, as we've discussed. The easiest atlas to use has two pages:

$$U_1 = \mathbb{C},\ \varphi_1 : U_1 \to V_1 = \mathbb{C},$$
$$\varphi_1(z) = z$$
$$U_2 = (\mathbb{C} \backslash \{0\}) \cup \{\infty\},\ \varphi_2 : U_2 \to V_2 = \mathbb{C}$$
$$\varphi_2(z) = \begin{cases} \dfrac{1}{z} & z \neq \infty \\ 0 & z = \infty \end{cases}$$

Then $U_1 \cap U_2$ is the punctured complex plane $\mathbb{C} \setminus \{0\}$ and
$\varphi_2 \circ \varphi_1^{-1}(z) = \dfrac{1}{z}$; $\varphi_1 \circ \varphi_2^{-1}(z) = \dfrac{1}{z}$.



Definition: Let S_1, S_2 be Riemann surfaces, and $f: S_1 \to S_2$ be a function. Then f is *analytic* if and only if each of the corresponding maps from atlas pages of S_1 to atlas pages of S_2 are analytic. Precisely, given an atlas for S_1:

$$\left\{U_\alpha, \varphi_\alpha : U_\alpha \to V_\alpha\right\}_{\alpha \in A}$$

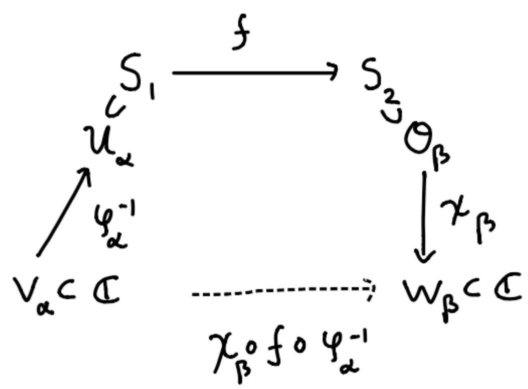
and and atlas for S_2

$$\left\{O_\beta, \chi_\beta : O_\beta \to W_\beta\right\}_{\beta \in B}$$

then f is defined to be analytic if and only if each triple composition

$$\chi_\beta \circ f \circ \varphi_\alpha^{-1} : V_\alpha \to W_\beta$$

is analytic.



So for a function $f: \mathbb{C} \to \mathbb{C} \cup \infty$ there are two cases to consider, in order to deduce whether f is analytic near z_0, as a map of Riemann surfaces:

$f(z_0) \in \mathbb{C}$: usual definition.

$f(z_0) \notin \mathbb{C}$ or undefined: Does $\dfrac{1}{f(z)}$ have a removable singularity at z_0? In other words does $f(z)$ have a pole at z_0, so that $f(z_0) = \infty$?

The text defined a *meromorphic function* on $\mathbb{C}$ to be one which is analytic except for a countable number of pole singularities. This corresponds to $f: \mathbb{C} \to \mathbb{C} \cup \infty$ being analytic as a function between Riemann surfaces.

For a function $f: \mathbb{C} \cup \infty \to \mathbb{C} \cup \infty$ there are two additional cases to consider to decide whether f is analytic as a function between Riemann surfaces:

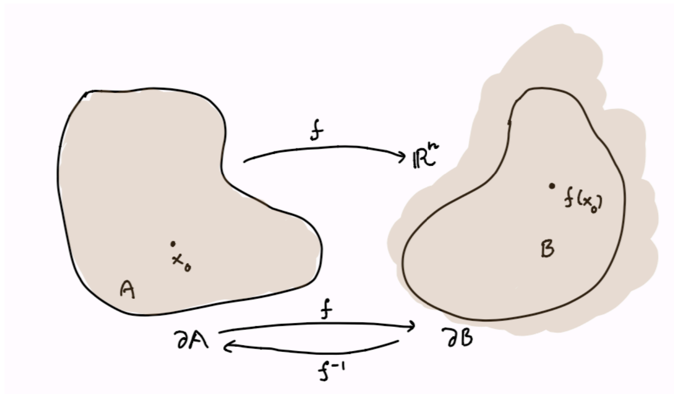
$z_0 = \infty, f(z_0) \in \mathbb{C}$: Does $f\left(\frac{1}{z}\right)$ have a removable singularity at $z = 0$?

$z_0 = \infty, f(z_0) \notin \mathbb{C}$: Does $\dfrac{1}{f\left(\frac{1}{z}\right)}$ have a removable singularity at $z = 0$?

Appendix: Magic Theorem Let $A, B \subseteq \mathbb{R}^n$ be open, connected, bounded sets.
Let $f: A \to \mathbb{R}^n$, $f \in C^1$, with $df_x : T_x\mathbb{R}^n \to T_{f(x)}\mathbb{R}^n$ invertible $\forall\ x \in A$ (i.e. the Jacobian matrix is invertible). Furthermore, assume

- $f: \overline{A} \to \mathbb{R}^n$ is continuous and one-to-one.
- $f(\delta A) = \delta B$
- $f(x_0) \in B$ for at least one $x_0 \in A$.

Then $f(A) = B$ and f is a global *diffeomorphism* between A and B. (i.e. $f^{-1} : B \to A$ is also differentiable), and $f^{-1}: \overline{B} \to \overline{A}$ is continuous.



proof: Step 1: $f(A) \subseteq B$.
proof: Let

$$\mathcal{O} := \{x \in A \,|\, f(x) \in B\}$$

Then

- $x_0 \in \mathcal{O}$
- $\mathcal{O}$ is open by the local inverse function theorem, since $x_1 \in \mathcal{O}$ and $f(x_1) \in B$ implies there is a local inverse function from an open neighborhood of $f(x_1)$ in B, back to a neighborhood of x_1 in A.
- $\mathcal{O}$ is closed in A because if $\{x_k\} \subseteq \mathcal{O}$, $\{x_k\} \to x \in A$ then $\{f(x_k)\} \to f(x)$ and since $\{f(x_k)\} \subseteq B$ we have $f(x) \in \overline{B}$. But since f is one-one and maps the boundary of A bijectively to the boundary of B, $f(x)$ cannot be in the boundary of B. Thus $f(x) \in B$.
- Thus, since A is connected, $\mathcal{O}$ is all of A, and $f(A) \subseteq B$.

Step 2: $f(A) = B$.
proof:

- $f(A)$ is open (by the local inverse function theorem again), so $f(A) \subseteq B$ is open.
- And $f(A)$ is closed in B because if

$$\{f(x_k)\} = \{y_k\} \subseteq f(A), \text{ with } \{y_k\} \to y \in B,$$

then because $\overline{A}$ is compact, a subsequence $\{x_{k_j}\} \to x \in \overline{A}$ with $\{f(x_{k_j})\} \to f(x) = y$, so $x \notin \delta A$ because $y \in B$, so $x \in A$ and $y \in f(A)$.

- So, because B is connected, $f(A)$ is all of B.

QED.

Remark: In $\mathbb{C}$ you can also imply this theorem to unbounded domains, i.e. in $\mathbb{C} \cup \{\infty\}$ because of the following diagram, in which $f_2 \circ f \circ f_1^{-1}$ satisfies the hypotheses of the original theorem: